Announcements

1) Take-home midterm up on Canvas, due Tuesday next week.

NO COLLABORATION!

2) In-Class midterm Tuesday next week

3) Colloquium tomorrow

4-5 CB 2062

"Planar Algebra"

Definition: (well/ill-conditioned)

A problem given by a function $f: V \to W$ where V and W are normed linear spaces is called well-conditioned if K is "small" relative to a given tolerance.

The problem is ill-conditioned if K is not small relative to the given tolerance.

Example 1 : (ill-conditioned)

$$f(x,y) = x - y \quad \text{(use } \infty\text{-norm)}$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix}$$

(original f is linear)

$$\|J\|_\infty = \text{maximal row sum}$$

$$= 2$$

$$K = \frac{\|J\|_\infty \|(x,y)\|_\infty}{\|f(x,y)\|_\infty}$$

$$= \frac{2\max\{|x|,|y|\}}{|f(x,y)|}$$

$$= \frac{2\max\{|x|,|y|\}}{|x-y|}$$

For $x = y$, this isn't even defined!

This says when you are "close" to the line $y = x$, small changes in y or x can produce huge changes in K. So this "problem" is ill-conditioned.

Gripe

Perturbing polynomial coefficients (ill-conditioned)

Text writes

$$\text{"} \delta x_j = \frac{-(\delta a_i) x_j^i}{p'(x_j)} \text{"}$$

where p is a polynomial, x_j is a root, and a_i is a coefficient of p

This is only an approximation, not an equality!

Example 2: polynomial

$$p(x) = 2x + 1$$

$$q(x) = (2.0001)x + 1$$

Jacobian of $p = 2$

Jacobian of $q = 2.0001$

$\delta a_1 = .0001$ (change in 1^{st} coordinate)

Root of $p = -1/2$

Root of $q = -1/2.0001$

Difference between these roots is $\delta x_1 = \delta x$

$$= \frac{.0001}{2(2.0001)}$$

$$\approx .00002499$$

Computing

$$\frac{-(\delta a_1)\,x}{p'(x)}$$

$$= \frac{(-.0001)\,x}{2} \quad \text{(x = root of p)}$$

$$= \frac{(-.0001)(-1/2)}{2}$$

$$= \frac{.0001}{4}$$

$$= .000025 \quad \text{exactly!}$$

So although

δx and $-\dfrac{(\delta a_1)x}{p'(x)}$

are very close,

they're not equal.

The proper statement is

$$\delta x_j \approx - \frac{(\delta a_i) x_j^i}{p'(x_j)}.$$

Note that the problem of finding roots of polynomials after perturbing a coefficint is ill-conditioned

The reason is that x_j could also be a root of p', again giving an undefined quotient!

Why is it true?

Let $x_1, x_2, \ldots, x_n$ be the roots of p, and let $a_0, a_1, \ldots, a_n$ be the coefficients of p. Fix $1 \le j \le n$, $0 \le i \le n$.

Then
$$p(x) = \sum_{k=0}^{n} a_k x^k$$
$$= a_n \prod_{j=1}^{n} (x - x_j)$$

Let $q(x) = \sum_{\substack{k=0 \\ k \neq i}}^{n} a_k x^k + (a_i + \delta a_i) x^i$

Let $y_j = x_j + \delta x_j$ be

the j^{th} root of q,

so $q(y_j) = 0$.

Appealing to the definition of the derivative,

$$p'(x_j) \approx \frac{p(x_j+\delta x_j) - \overbrace{p(x_j)}^{=0}}{\delta x_j}$$

$$= \frac{p(y_j)}{\delta x_j},$$

so

$$\delta x_j = \frac{p(y_j)}{p'(x_j)}.$$

Now since $q(y_j) = 0$,

$$p(y_j) = p(y_j) - q(y_j)$$

$$= -\delta a_i \, y_j^i .$$

by using the fact that the coefficients of p and q are identical except for the i^{th} coefficient.

Then

$$\delta x_j = \frac{p(y_j)}{p'(x_j)}$$

$$= \frac{-\delta a_i y_j^i}{p'(x_j)}$$

$$= \frac{-\delta a_i (x_j + \delta x_j)^i}{p'(x_j)}$$

$$\approx \frac{-\delta a_i x_j^i}{p'(x_j)}$$

Conditioning and Matrices

Invertibility

Since the Jacobian of any matrix is simply the matrix itself, if $A \in \mathbb{C}^{m \times n}$,

$$\boxed{\kappa = \frac{\|A\| \, \|x\|}{\|Ax\|}}$$

If $m=n$ and A is invertible,

$$\|x\| = \|(A^{-1} \cdot A)x\|$$

$$\leq \|A^{-1}\| \|Ax\| .$$

Therefore,

$$K = \frac{\|A\| \, \|x\|}{\|Ax\|}$$

$$\leq \frac{\|A\| \, \|A^{-1}\| \, \cancel{\|Ax\|}}{\cancel{\|Ax\|}}$$

$$= \|A\| \, \|A^{-1}\| .$$

Q: Is this the best possible bound?

A: Yes! By choosing x to be a right singular vector in the SVD of A where x corresponds to the minimal singular value of A.